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# МОДЕЛИРОВАНИЕ СОХРАНЕНИЯ ЭНЕРГИИ В ПОЛНОСТЬЮ ИНТЕГРИРУЕМОЙ СИСТЕМЕ БУССИНЕСКА

Представлен вывод плотности энергии и потока энергии морских волн в рамках модели Каупа, являющейся одним из вариантов системы Буссинеска. Этот вывод основан на процедуре восстановления поля скорости и давления в объеме жидкости под свободной поверхностью, обобщающей метод, ранее предложенный авторами. Показано, что полная волновая энергия совпадает с гамильтонианом, представленным В.Крейгом и М.Грувсом.

**Ключевые слова:** нелинейные волны на воде, энергия волн.

The object of this note is the derivation of expressions for the energy density and the energy flux associated to an integrable system of evolution equations modeling the propagation of weakly nonlinear long waves on the surface of an incompressible, inviscid fluid. In particular, the focus is on the so-called Kaup system

$$\begin{aligned} \eta_t + h_0\omega_x + (\eta\omega)_x + \frac{h_0^3}{3}\omega_{xxx} = 0, \\ \omega_t + g\eta_x + \omega\omega_x = 0, \end{aligned} \tag{1}$$

where $\eta(x, t)$ describes the displacement of the free surface from the rest position at a point $x$ and time $t$, $\omega(x,t)$ is the horizontal velocity of the fluid at the free surface, $g$ represents the gravitational acceleration, and $h_0$ denotes the undisturbed depth of the fluid.

The Kaup system belongs to the family of long-wave models developed by Boussinesq [3], extended by Peregrine [4] and Nwogu [5], and represents a special case of the general system of long-wave models derived recently in [6]. These models describe the propagation of surface gravity waves of small amplitude and long wavelength in a horizontal channel of uniform depth, and under the assumption that the flow is irrotational, and essentially two-dimensional. Even though this system is not well posed in the Hadamard sense [7], it is important because it has an integrable Hamiltonian structure [8]. Moreover, (1) appears naturally when the derivation of the long-wave system is based on approximating the Hamiltonian function of the full surface water-wave problem. As explained in detail in [2], this formulation is given in terms of the evaluation of the velocity potential at the free surface, and relies on the Hamiltonian structure of the full water wave problem found by Zakharov [9]. A version of this system also appears in the context of interfacial waves if it is required that an approximate Hamiltonian function be conserved [10]. In the context of surface waves, the Hamiltonian function is given by

$$H = \rho\int_{-\infty}^{\infty}\left\{\frac{g}{2}\eta^2 + \frac{1}{2}(h_0+\eta)\omega^2 - \frac{h_0^3}{6}\omega_x^2\right\}dx \tag{2}$$

and the system (1) is the corresponding Hamiltonian dynamical system. As shown in [8], the system (1) is actually a completely integrable dynamical system.

Since the derivation of (1) presented in [2] is based on approximating the total energy of the fluid system governed by the Euler equations, the integral (2) appears naturally as the total energy of the system in the approximation leading to (1). On the other hand, for some purposes it can also be of value to have knowledge of the energy contained in a smaller section of the fluid, and in particular of the local energy flux. These quantities can be important in various situations of practical significance, such as the modeling of tsunamis [11], undular bores [12, 13], and in the analysis of shoaling waves [14]. It is the purpose of the present paper to shed light on the conservation of energy in the context of the Kaup system, and to provide expressions for the local energy flux and density in terms of a solution $\eta$, $\omega$ of (1).

The derivation of Boussinesq models for surface water waves is based on the assumption that there is an approximate balance between nonlinear steepening effects and dispersive spreading [6, 12]. Supposing that $a$ is a representative amplitude, and $\ell$ represents a dominant wavelength, the Boussinesq scaling regime describes waves for which the two small parameters $\alpha = a/h_0 << 1$ and $\beta = h_0^2/\ell^2 << 1$ are approximately equal. The system (1) consists of an approximate mass balance equation (first equation), and an approximate horizontal momentum balance equation (second equation), and is obtained by neglecting terms of order $\alpha\beta$ and $\beta^2$. In order to obtain a corresponding approximation for the energy density $E$ and energy flux $q_E$, one may stipulate that the differential energy balance equation

$$\frac{\partial}{\partial t}E + \frac{\partial}{\partial x}q_E = 0$$

hold to the same order in $\alpha$ and $\beta$ as the evolution equations.

A related development was presented in [15] for the single second-order Boussinesq equation. However, the expressions found in [15] are quite different from the formulas developed here, and the method is based on requiring exact energy conservation instead of approximate conservation. In [1], a method was developed to derive mass, momentum and energy densities and fluxes associated to the family of Boussinesq systems found in [6]. The expressions were then compared to the corresponding quantities in the shallow-water theory. In the present note, we use the same method, but a different definition for the potential energy in order to show that the Hamiltonian function associated to (1) represents exactly the total energy of the system. We expect that similar considerations will yield information about the energy density and flux for internal waves modeled by the Kaup equation [10], but the details have not been carried out yet.

**Preliminaries.** The surface water-wave problem is generally described by the Euler equations with no-flow conditions at the bottom, and kinematic and dynamic boundary conditions at the free surface. Since the fluid is incompressible and the flow is assumed irrotational, the problem may be formulated in terms of the velocity potential $\varphi(x, z, t)$ and the surface elevation $\eta(x, t)$. The pressure is eliminated with help of the Bernoulli equation, the potential $\varphi$ satisfies Laplace's equation, and the free-surface boundary conditions are formulated in terms of the potential and the surface excursion by

$$\left.\begin{aligned} \eta_t + \varphi_x\eta_x - \varphi_z &= 0, \\ \varphi_t + \frac{1}{2}\left(\varphi_x^2 + \varphi_z^2\right) + g\eta &= 0, \end{aligned}\right\} \text{ on } \; z = \eta(x,t).$$

To set the stage for approximating the energy density and flux, we briefly recall the derivation of the Kaup system. In order to identify the relevant terms in the equations, the variables are non-dimensionalized in the following way:

$$\tilde{x} = \frac{x}{\ell}, \quad \tilde{z} = \frac{z + h_0}{h_0}, \quad \tilde{\eta} = \frac{\eta}{a}, \quad \tilde{t} = \frac{c_0 t}{\ell}, \quad \tilde{\varphi} = \frac{c_0}{ga\ell}\varphi,$$

where $c_0 = \sqrt{gh_0}$ . The free-surface boundary conditions then take the form

$$\left.\begin{aligned} &\tilde{\eta}_{\tilde{t}} + \alpha\tilde{\varphi}_{\tilde{x}}\tilde{\eta}_{\tilde{x}} - \frac{1}{\beta}\tilde{\varphi}_{\tilde{z}} = 0, \\ &\tilde{\eta} + \tilde{\varphi}_{\tilde{t}} + \frac{1}{2}\left(\alpha\tilde{\varphi}_{\tilde{x}}^2 + \frac{\alpha}{\beta}\tilde{\varphi}_{\tilde{z}}^2\right) = 0, \end{aligned}\right\} \text{ on } \tilde{z} = 1 + \alpha\tilde{\eta}. \tag{3}$$

The standard approach consists of developing the potential $\varphi$ in an asymptotic series, and using the Laplace equation and Neumann boundary condition at the bottom to write the non-dimensional velocity potential $\tilde{\varphi}$ in the form

$$\tilde{\varphi} = \tilde{f} - \frac{\tilde{z}^2}{2}\tilde{f}_{\tilde{x}\tilde{x}}\beta + O(\beta^2). \tag{4}$$

Substituting this expression into the second boundary condition at the free surface yields the relation

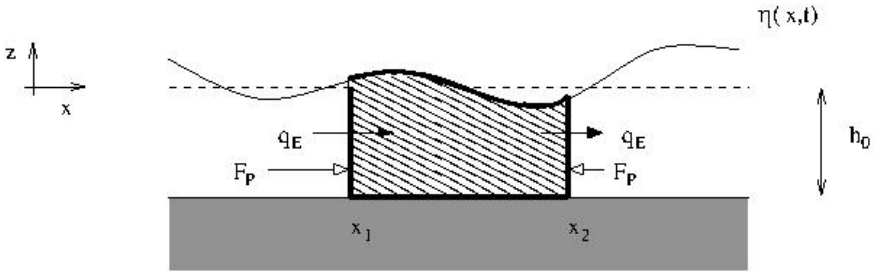
$$\tilde{\eta} + \tilde{f}_{\tilde{t}} - \frac{\beta}{2}\tilde{f}_{\tilde{x}\tilde{x}\tilde{t}} + \frac{\alpha}{2}\tilde{f}_{\tilde{x}}^2 = O(\alpha\beta, \beta^2). \tag{5}$$

To find a closed system of two evolution equations, we insert the asymptotic expression for $\tilde{\varphi}$ in the first equation in (3), and collect all terms of zeroth and first order in α and β. Then, we diferentiate (5), and express the equations in terms of the non-dimensional horizontal velocity at the bottom $f_{\tilde{x}} = \tilde{v}$ . This procedure yields the equations

$$\left.\begin{aligned} &\tilde{\eta}_{\tilde{t}} + \tilde{v}_{\tilde{x}} + \alpha(\tilde{\eta}\tilde{v})_{\tilde{x}} - \frac{1}{6}\beta\tilde{v}_{\tilde{x}\tilde{x}\tilde{x}} = O(\alpha\beta, \beta^2), \\ &\tilde{\eta}_{\tilde{x}} + \tilde{v}_{\tilde{t}} - \frac{1}{2}\beta\tilde{v}_{\tilde{x}\tilde{x}\tilde{t}} + \alpha\tilde{v}\tilde{v}_{\tilde{x}} = O(\alpha\beta, \beta^2). \end{aligned}\right. \tag{6}$$

Now if $\tilde{\omega}$ denotes the nondimensional velocity at a nondimensional height $\tilde{z} = 1$ in the fluid column, then $\tilde{v}$ may be expressed in terms of $\tilde{\omega}$ as

$$\tilde{v} = \tilde{\omega} + \frac{1}{2}\beta\tilde{\omega}_{\tilde{x}\tilde{x}} + O(\beta^2). \tag{7}$$




The schematic elucidates the geometric setup of the problem. The undisturbed water depth is $h_0$ , and the *x*-axis is aligned with the free surface at rest. The free surface is described by a function η(*x*, *t*). The figure shows a control interval delimited by $x_1$ and $x_2$ on the abscissa, and the arrows indicate pressure force $F_P$ and energy flux $q_E$ .

Writing (6) in terms of $\tilde{\omega}$, the Kaup system appears in the non-dimensional form

$$\tilde{\eta}_{\tilde{t}} + \tilde{\omega}_{\tilde{x}} + \alpha(\tilde{\eta}\tilde{\omega})_{\tilde{x}} + \frac{1}{3}\beta\tilde{\omega}_{\tilde{x}\tilde{x}\tilde{x}} = O(\alpha\beta,\beta^2),$$
$$\tilde{\eta}_{\tilde{x}} + \tilde{\omega}_{\tilde{t}} + \alpha\tilde{\omega}\tilde{\omega}_{\tilde{x}} = O(\alpha\beta,\beta^2).$$

If terms of order $O(\alpha\beta,\beta^2)$ are disregarded, and dimensional variables are used, the Kaup system (1) is recovered.

As the aim here is to find expressions for the energy density and flux, it is essential to derive an approximation for the pressure associated to the (1), and we follow the method laid down in [1]. With the help of the Bernoulli equation, the dynamic pressure can be written as

$$P' = P + \rho gz = -\rho\varphi_t - \frac{\rho}{2}|\nabla\varphi|^2,$$

where atmospheric pressure has been set as the reference pressure, and ρ denotes the density of the fluid. Converting to non-dimensional variables, using the asymptotic expansion for the velocity potential (4), and using the identity (5), the second-order dynamic pressure emerges in the form

$$\tilde{P}' = \tilde{\eta} + \frac{1}{2}\beta(\tilde{z}^2 - 1)\tilde{\omega}_{\tilde{x}\tilde{t}} + O(\alpha\beta,\beta^2).$$

Converting to dimensional variables, it may be confirmed that this agrees with the expression for the pressure associated to the so-called «classical» Boussinesq system given in [16]. Similar considerations as in [1] were also used in [17] to derive the pressure associated to the Green-Naghdi system which models higher-order effects in surface waves.

**Energy balance.** In order to derive the energy density and flux we consider the total mechanical energy inside a control volume in the fluid and above the interval [$x_1$, $x_2$], as shown in Fig.1. The total energy in this control volume is

$$\varepsilon = \frac{1}{2}\int_{x_1}^{x_2}\int_{-h_0}^{\eta}\rho|\nabla\varphi|^2\,dzdx + \int_{x_1}^{x_2}\int_{0}^{\eta}\rho gz\,dzdx,$$

where the first term represents the kinetic energy, and the second term the potential energy. The potential energy has been normalized in such a way that a particle located at the level of the undisturbed free surface has zero potential energy, and such that the total potential energy is zero when no wave motion is present. Following [18], conservation of total mechanical energy is written as

$$\frac{d}{dt}\int_{x_1}^{x_2}\int_{-h_0}^{\eta}\frac{\rho}{2}|\nabla\varphi|^2\,dzdx + \frac{d}{dt}\int_{x_1}^{x_2}\int_{0}^{\eta}\rho gz\,dzdx = \left[\int_{-h_0}^{\eta}\left\{\frac{\rho}{2}|\nabla\varphi|^2 + \rho gz\right\}\varphi_x dz + \int_{-h_0}^{\eta}\varphi_x P dz\right]_{x_2}^{x_1}.$$

We convert to non-dimensional variables, and compute the left and right hand sides individually. Substituting the expressions for $\tilde{\varphi}_{\tilde{x}}$ and $\tilde{\varphi}_{\tilde{z}}$, recalling that $\tilde{f}_{\tilde{x}} = \tilde{v}$ is the velocity at the bottom, and using (7), there appears

$$\frac{d}{d\tilde{t}}\int_{x_1/\ell}^{x_2/\ell}\int_{0}^{1+\alpha\tilde{\eta}}\left\{\frac{\alpha^2}{2}\left(\tilde{\varphi}_{\tilde{x}}^2 + \frac{1}{\beta}\tilde{\varphi}_{\tilde{z}}^2\right)\right\}d\tilde{z}\,d\tilde{x} + \frac{d}{dt}\int_{x_1/\ell}^{x_2/\ell}\int_{1}^{1+\alpha\tilde{\eta}}(\tilde{z}-1)d\tilde{z}\,d\tilde{x} =$$
$$= \frac{d}{d\tilde{t}}\int_{x_1/\ell}^{x_2/\ell}\left\{\frac{\alpha^2}{2}\left(\tilde{f}_{\tilde{x}}^2(1+\alpha\tilde{\eta}) - \frac{\beta}{3}\tilde{f}_{\tilde{x}}\tilde{f}_{\tilde{x}\tilde{x}\tilde{x}} + \frac{\beta}{3}\tilde{f}_{\tilde{x}\tilde{x}}^2\right) + \frac{\alpha^2}{2}\tilde{\eta}^2\right\}d\tilde{x} + O\left(\alpha^3\beta,\alpha^2\beta^2\right) =$$
$$= \frac{d}{d\tilde{t}}\int_{x_1/\ell}^{x_2/\ell}\left\{\frac{\alpha^2}{2}(1+\alpha\tilde{\eta})\tilde{\omega}^2 + \frac{\alpha^2\beta}{3}\tilde{\omega}\tilde{\omega}_{\tilde{x}\tilde{x}} + \frac{\alpha^2\beta}{6}\tilde{\omega}_{\tilde{x}}^2 + \frac{\alpha^2}{2}\tilde{\eta}^2\right\}d\tilde{x} + O\left(\alpha^3\beta,\alpha^2\beta^2\right).$$

After integrating and using the substitution for $\tilde{f}_{\tilde{x}}$, the energy flux and work done by the pressure force on the right hand side of the energy balance reduce to

$$\int_0^{1+\alpha\tilde{\eta}} \frac{\alpha^3}{2}\left(\tilde{\varphi}_{\tilde{x}}^3 + \frac{1}{\beta}\tilde{\varphi}_{\tilde{z}}^2\tilde{\varphi}_{\tilde{x}}\right)d\tilde{z} + \alpha^2\int_0^{1+\alpha\tilde{\eta}} \tilde{P}'\tilde{\varphi}_{\tilde{x}}\, d\tilde{z} =$$

$$= \int_0^{1+\alpha\tilde{\eta}} \left\{\frac{\alpha^3}{2}\,\tilde{f}_{\tilde{x}}^3 + \alpha^2\left(\,\tilde{f}_{\tilde{x}}\,\tilde{\eta} - \frac{\tilde{z}^2}{2}\,\tilde{f}_{\tilde{x}\tilde{x}\tilde{x}}\tilde{\eta}\beta + \frac{\beta}{2}\left(\tilde{z}^2-1\right)\tilde{\omega}_{\tilde{x}\tilde{t}}\,\tilde{f}_{\tilde{x}}\right)\right\}d\tilde{z} + \mathrm{O}\left(\alpha^3\beta, \alpha^2\beta^2\right) =$$

$$= \frac{\alpha^3}{2}\,\tilde{f}_{\tilde{x}}^3 + \alpha^2\,\tilde{f}_{\tilde{x}}\,\tilde{\eta}\left(1+\alpha\tilde{\eta}\right) - \frac{\alpha^2\beta}{6}\,\tilde{f}_{\tilde{x}\tilde{x}\tilde{x}}\tilde{\eta} - \frac{\alpha^2\beta}{3}\tilde{\omega}_{\tilde{x}\tilde{t}}\,\tilde{f}_{\tilde{x}} + \mathrm{O}\left(\alpha^4, \alpha^3\beta, \alpha^2\beta^2\right) =$$

$$= \frac{\alpha^3}{2}\tilde{\omega}^3 + \alpha^2\tilde{\omega}\tilde{\eta}\left(1+\alpha\tilde{\eta}\right) + \frac{\alpha^2\beta}{3}\tilde{\eta}\tilde{\omega}_{\tilde{x}\tilde{x}} - \frac{\alpha^2\beta}{3}\tilde{\omega}\tilde{\omega}_{\tilde{x}\tilde{t}} + \mathrm{O}\left(\alpha^4, \alpha^3\beta, \alpha^2\beta^2\right).$$

Thus the energy balance equation transforms to

$$\frac{d}{d\tilde{t}}\int_{x_1/\ell}^{x_2/\ell}\left\{\frac{\alpha^2}{2}\,\tilde{\eta}^2 + \frac{\alpha^2}{2}\left(1+\alpha\tilde{\eta}\right)\tilde{\omega}^2 + \frac{\alpha^2\beta}{3}\,\tilde{\omega}\tilde{\omega}_{\tilde{x}\tilde{x}} + \frac{\alpha^2\beta}{6}\,\tilde{\omega}_{\tilde{x}}^2\right\}d\tilde{x} =$$

$$= \left[\frac{\alpha^3}{2}\tilde{\omega}^3 + \alpha^2\tilde{\omega}\tilde{\eta}\left(1+\alpha\tilde{\eta}\right) + \frac{\alpha^2\beta}{3}\,\tilde{\eta}\tilde{\omega}_{\tilde{x}\tilde{x}} - \frac{\alpha^2\beta}{3}\,\tilde{\omega}\tilde{\omega}_{\tilde{x}\tilde{t}}\right]_{x_2}^{x_1} + \mathrm{O}\left(\alpha^4, \alpha^3\beta, \alpha^2\beta^2\right). \tag{8}$$

A differentiation in *x* finally yields the differential energy balance equation in the form

$$\frac{\partial}{\partial\tilde{t}}\,\tilde{E} + \frac{\partial}{\partial\tilde{x}}\,\tilde{q}_E = \mathrm{O}\left(\alpha^2, \alpha\beta, \beta^2\right). \tag{9}$$

Taking the appropriate terms in the energy density and flux in (8) which are of order zero or one in the differential energy balance (9), we find the non-dimensional energy density to be

$$\tilde{E} = \frac{\alpha^2}{2}\,\tilde{\eta}^2 + \frac{\alpha^2}{2}\left(1+\alpha\tilde{\eta}\right)\tilde{\omega}^2 + \frac{\alpha^2\beta}{3}\,\tilde{\omega}\tilde{\omega}_{\tilde{x}\tilde{x}} + \frac{\alpha^2\beta}{6}\,\tilde{\omega}_{\tilde{x}}^2.$$

The non-dimensional energy flux (corrected for the work done by pressure force) is given by

$$\tilde{q}_E = \frac{\alpha^3}{2}\tilde{\omega}^3 + \alpha^2\tilde{\omega}\tilde{\eta}\left(1+\alpha\tilde{\eta}\right) + \frac{\alpha^2\beta}{3}\,\tilde{\eta}\tilde{\omega}_{\tilde{x}\tilde{x}} - \frac{\alpha^2\beta}{3}\,\tilde{\omega}\tilde{\omega}_{\tilde{x}\tilde{t}}.$$

Using the natural scalings $E = \rho h_0 c_0^2 \tilde{E}$ and $\tilde{q}_E = \rho h_0 c_0^3 \tilde{q}_E$, the dimensional forms of the energy density and energy flux are obtained in the form

$$E = \rho\left\{\frac{g}{2}\eta^2 + \frac{1}{2}\left(h_0+\eta\right)\omega^2 + \frac{h_0^3}{3}\,\omega\omega_{xx} + \frac{h_0^3}{6}\,\omega_x^2\right\},$$

$$q_E = \rho\left\{\frac{h_0}{2}\,\omega^3 + g\eta\omega\left(h_0+\eta\right) + \frac{gh_0^3}{3}\,\eta\omega_{xx} - \frac{h_0^3}{3}\,\omega\omega_{xt}\right\}.$$

In particular, $q_E(x,t)$ gives the energy flux due to the wave motion at a point *x* and a time *t*. Integrating $E(x, t)$ over an interval $[x_1, x_2]$ yields the energy due to the wave motion in the control interval shown in Figure (1) at a time *t*, and to the same order of approximation as the system (1) is valid. If the surface disturbance is localized, so that $\eta$ and $\omega$ decay to zero at

infinity, and the integration of $E$ is taken over the entire real line, an integration by parts shows that the total energy is equal to the Hamiltonian function derived by Craig and Groves [2]:

$$H = \int_{-\infty}^{\infty} E(x,t)dx.$$

**Conclusions.** Using the method developed in [1], expressions for energy flux and density per unit span in the transverse direction have been found. These quantities are given in terms of the principal unknowns η and ω of the Boussinesq system. Integrating $E$ over the real line shows that the total energy in this description is equal to the Hamiltonian function used by Craig and Groves [2]. This result may be viewed as a further validation of the method presented in [1]. We expect that similar considerations can be used to understand energy balances in internal wave models, such as in the equations studied in [10].

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