УДК 551.465

© *Н. В. Журбас*
Институт океанологии им. П. П. Ширшова РАН, Москва
nvzhurbas@gmail.com

О СПЕКТРАХ СОБСТВЕННЫХ ЗНАЧЕНИЙ В МОДЕЛЬНОЙ ЗАДАЧЕ ОПИСАНИЯ ОБРАЗОВАНИЯ КРУПНОМАСШТАБНЫХ ИНТРУЗИЙ В АРКТИЧЕСКОМ БАССЕЙНЕ

Статья поступила в редакцию 29.11.2017, после доработки 23.01.2018.

Рассчитываются спектры собственных значений в модельной задаче исследования неустойчивости геострофического течения с линейным вертикальным сдвигом скорости в ограниченном по вертикали слое. В модельном уравнении учитывается влияние на динамику устойчивых и неустойчивых возмущений течения вертикальной диффузии плавучести. Задача сводится к численному решению несамосопряженного дифференциального уравнения третьего порядка с малым параметром при старшей производной при граничных условиях, типичных для океана. Решение ищется в виде степенного ряда. Поиск собственных значений приводит к поиску корней полинома высокой степени. Представлены спектры собственных значений для различных значений безразмерного параметра задачи. Результаты расчетов собственных значений сравниваются с результатами, полученными альтернативным методом решения эквивалентной задачи. Обращается внимание, что рассмотренная неустойчивость течения является осцилляционной неустойчивостью, которая кардинально отличается от типичной монотонной неустойчивости фронтов на масштабах интрузионного расслоения повсюду за исключением экваториальной зоны. Полученные результаты важны для анализа механизмов образования интрузий в Арктическом бассейне, наблюдающихся в условиях абсолютно устойчивой стратификации, т. е. когда с увеличением глубины уменьшается температура и увеличивается соленость.

Ключевые слова: неустойчивость фронтальных разделов, интрузионное расслоение, Арктический бассейн.

N. V. Zhurbas
Shirshov Institute of Oceanology, Russian Academy of Sciences, Moscow, Russia

ON THE EIGENVALUE SPECTRA FOR A MODEL PROBLEM DESCRIBING FORMATION OF THE LARGE-SCALE INTRUSIONS IN THE ARCTIC BASIN

Received 29.11.2017, in final form 23.01.2018.

To study the geostrophic flow instability with a linear vertical velocity shear in a vertically bounded layer the eigenvalue problem solution is considered. The vertical buoyancy diffusion effect on stable and unstable flow perturbations dynamics is taken into account in the model equation. The problem is reduced to a numerical solution of a non-self-adjoint third order differential equation with a small parameter at the highest derivative under the boundary conditions typical for the ocean. The solutions are sought as the power expansions at zero. The eigenvalues calculation leads to the search for the roots of a high-degree polynomial. The eigenvalue spectra are presented for various values of the problem dimensionless parameter. The results of the eigenvalues calculation are compared with the results obtained by an alternative method for solving an equivalent problem. It is noteworthy that the flow instability examined is an oscillatory instability, which is fundamentally different from the typical monotonous instability of fronts on the scales of the intrusion formation everywhere except for the equatorial zone. The results obtained are significant for analyzing the mechanisms of intrusive layers formation, which occur in the Arctic basin under conditions of absolutely stable stratification (decrease of the mean temperature with depth is accompanied by increase of the mean salinity).

Key words: instability of fronts, intrusive layering, Arctic Basin.

Ссылка для цитирования: *Журбас Н. В.* О спектрах собственных значений в модельной задаче описания образования крупномасштабных интрузий в Арктическом бассейне // Фундаментальная и прикладная гидрофизика. 2018. Т. 11, № 1. С. 40—45.

For citation: *Zhurbas N. V.* On the eigenvalue spectra for a model problem describing formation of the large-scale intrusions in the Arctic basin. *Fundamentalnaya I Prikladnaya Gidrofizika.* 2018, 11, 1, 40—45.

DOI: 10.7868/S2073667318010045

Investigation of the intrusive layering mechanisms in the ocean is important for parametrization of the exchange and mixing processes, for estimating the changes in the heat and salt contents of different water masses, and for forecasting the effect of climatic changes on the structural features of ocean waters at various depths (see, for example, [1—3]). Since interleaving is the characteristic feature of the Arctic basin that can be observed at different spatial and temporal scales [4, 5], the description of the intrusions generating processes based on mathematical models becomes very relevant, especially because it can allow to determine physical causes of instability of currents and fronts. In [6, 7] using analytical solutions accounting for diffusion of buoyancy it was found that long-wave perturbations of the geostrophic flow with a linear vertical velocity shear can be unstable (growing with time), and the phase velocity of unstable disturbances is directed along the mean flow and exceeds its maximum velocity. Therefore the instability obtained can be attributed neither to baroclinic instability nor instability of the critical layer.

The conclusion about the existence of a new type of long-wave perturbations instability [6, 7] was confirmed by the solution of the model problem on stable and unstable perturbations of the geostrophic flow in a vertically bounded layer with allowance for small but finite vertical circulations arising due to friction, beta effect and temporal variability of relative vorticity [8]. The problem was solved numerically by means of a high-precision method for solving fourth-order equations, which was proposed in the papers [9, 10] for calculating the spectra of the Orr-Sommerfeld problem. This method, based on power series expansions of the solution at the boundary and central points of the layer and on the joining of these expansions at an interior point, has been modified by taking boundary conditions specific to the ocean [11], which differ from the boundary conditions of the Orr-Sommerfeld problem and include the sought spectral parameter as a multiplier. Numerical calculations were validated using asymptotic estimates and comparing the eigenvalue dependencies on the model parameters, which were obtained by numerical and analytical modelling presented in [6, 7]. But taking into account that the instability of long-wave perturbations of the geostrophic flow with a linear vertical velocity shear is a fundamentally new effect, that is significant for understanding the mechanisms of large-scale intrusion formation in the Arctic basin, it seems reasonable to search for an additional clear evidence of numerical calculations [8, 11] reliability. The present work is concerned with the next topics: a) solving a model problem, equivalent to the problem [8], by means of a simple method alternative to the method [8—11], and comparison of the eigenvalues obtained by different methods; b) analysis of the eigenvalue spectra with a reference to description of intrusive layering in the Arctic basin.

Model setup, solution method and eigenvalue spectra. Stable and unstable perturbations of the geostrophic flow with a linear vertical shear of velocity in a vertically bounded layer can be described by the following dimensionless equation:

$$\left(1 - z^2 - c\right)\left(\frac{d^2F}{dz^2} - Bu\cdot\left(k^2+\pi^2\right)F\right) + \mu^2\cdot F + 2F = \frac{1}{ikR}\left(\frac{d^4F}{dz^4} - Pr\cdot Bu\cdot\left(k^2+\pi^2\right)\frac{d^2F}{dz^2}\right), \quad (1)$$

where $F(z)$ — perturbation of pressure, k — dimensionless wave number along the flow, $R = Pe{\cdot}H/L$, Pe — the Peclet number (an analog of the Reynolds number), H — layer thickness, L — transversal flow scale, $Bu = \left(HN_0 / Lf\right)^2$ — the Burger number, N_0 — is the buoyancy frequency, Pr — the Prandtl number, $\mu^2 = \beta\cdot N_0^{\ 2} / \left(sf^2\right)$ — a parameter characterizing the influence of beta effect on perturbation dynamics compared with the effect of linear velocity shear s [8], $c = c_1 + ic_2$ — complex phase velocity. Perturbation is unstable (growing with time) when Im$c = c_2 > 0$. The boundary conditions for equation (1) are the conditions for absence of vertical velocity at the layer boundaries:

$$-c\frac{dF}{dz} + 2zF = \frac{1}{ikR}\frac{d^3F}{dz^3}, \; z = \pm 1, \quad (2)$$

and zero flow of buoyancy at these boundaries:

$$\frac{d^2F}{dz^2} = 0, \; z = \pm 1. \quad (3)$$

The derivation of the presented model problem is described in detail in [6—8].

If the cross-frontal length scale and the length scale of perturbations exceed greatly the local Rossby radius, that is the condition $\delta^2 = Bu{\cdot}(k^2 + \pi^2) << 1$ is fulfilled, and parameter μ^2 has the same order of magnitude as δ^2, the problem (1)—(3) can be simplified at Pr ~ 1 by rewriting it in the following form (for more details, see [8]):

$$\left(1 - z^2 - c\right)\frac{dF_1}{dz} + 2zF_1 = \frac{1}{ikR}\frac{d^3F_1}{dz^3}, \tag{4}$$

$$\int_{-1}^{1}\left(-z^2 - c\right)F_1\left(z\right)dz - \frac{1}{ikR}\Pr\left(\left.\frac{dF_1}{dz}\right|_{z=1} - \left.\frac{dF_1}{dz}\right|_{z=-1}\right) = 0, \tag{5}$$

$$\frac{d^2F_1}{dz^2} = 0,\ z = \pm 1, \tag{6}$$

where F_1 is the first term in the expansion of the function F into a series in the parameter δ^2.

Equation (4) and conditions (5) and (6) describe perturbations of the geostrophic flow in a vertically bounded layer with allowance for small but finite vertical circulations arising due to friction, beta effect and temporal variability of relative vorticity. The model equation and the condition for absence of vertical velocity at the layer boundaries written in a form

$$\left(1 - z^2 - c\right)\frac{d^2F_1}{dz^2} + 2F_1 = \frac{1}{ikR}\frac{d^4F_1}{dz^4}, \tag{7}$$

$$-c\frac{dF_1}{dz} + 2zF_1 = \frac{1}{ikR}\frac{d^3F_1}{dz^3},\ z = \pm 1, \tag{8}$$

together with condition (6) define a problem equivalent to the problem (4)—(6) in the case of odd eigenfunctions. In [8] the calculation of eigenvalues for odd functions was carried out on the basis of the equation (7) and conditions (6), (8). We will consider the problem (4)—(6) for calculation the spectra of eigenvalues.

Considering that equation (4) does not change when z is replaced by $-z$ and $F_1(z)$ by $F_1(-z)$ or $F_1(z)$ by $-F_1(-z)$, for any c it necessarily has an even and odd solutions, which are linearly independent. Therefore the general solution of (4) can be written as

$$F_1 = A_1 \cdot F_{11} + A_2 \cdot F_{12} + A_3 \cdot F_{13}. \tag{9}$$

Let F_{11} and F_{12} be even and odd functions respectively. Then for the function F_{13} the following expression is true (see, for example, [12]):

$$F_{13} = F_{11} \cdot \int F_{12} \cdot \varphi \cdot dz - F_{12}\int F_{11}\varphi \cdot dz;\ \varphi = 1/(F_{12} \cdot dF_{11}/dz - F_{11} \cdot dF_{12}/dz)^2.$$

This implies that F_{13} is an even function. Thus the general solution of equation (4) consists of one odd and two even functions for any parameter c. For comparison with the results of calculating the eigenvalues obtained in [8] we will confine ourselves to calculations of the eigenvalues for odd solutions. Taking into account that the solutions of (4) and (7) are continuously differentiable functions (see, for example, [13]), we will seek the solution of (4) in the form of a power series:

$$F_{12}\left(z\right) = a_1 \cdot z + a_3 \cdot z^3 + a_5 \cdot z^5 + ... = \sum_{n=0}^{\infty} a_{2n+1} \cdot z^{2n+1}. \tag{10}$$

The series (10) converges for any $z < \infty$. Substituting (10) into equation (4), one can find recurrence relations for determining the coefficients in the expansion (10):

$$a_1 - c \cdot a_1 = 6 \cdot a_3 / ikR$$

$$\left(2n+1\right)\left(1-c\right)a_{2n+1} - \left(2n-1\right)a_{2n-1} + 2a_{2n-1} = \left(2n+3\right)\left(2n+2\right)\left(2n+1\right)a_{2n+3} / ikR,\ n = 1, 2, 3... \tag{11}$$

Setting $a_1 = 1$ in (11) without loss of generality we can express all the coefficients of the power series (10) in terms of the complex phase velocity c and the parameter of the problem kR.

The function (10) with the coefficients found on the basis of recurrence relations (11) will be an eigenfunction of problem (4)—(6) only if it satisfies conditions (5) and (6). It is easy to see that any odd function satisfies condition (5). Thus to determine the eigenvalues spectrum of the problem it is necessary and sufficient to find such values of parameter c for which the condition $F''_{12}\left(1\right) = 0$ is satisfied, since $F''_{12}\left(z\right)$ is an odd function. The condition $F''_{12}\left(1\right) = 0$ reduces to the problem of determining the roots of a polynomial $P\left(c\right) = \sum_{n=0}^{\infty} b_n \cdot c^n = 0$,

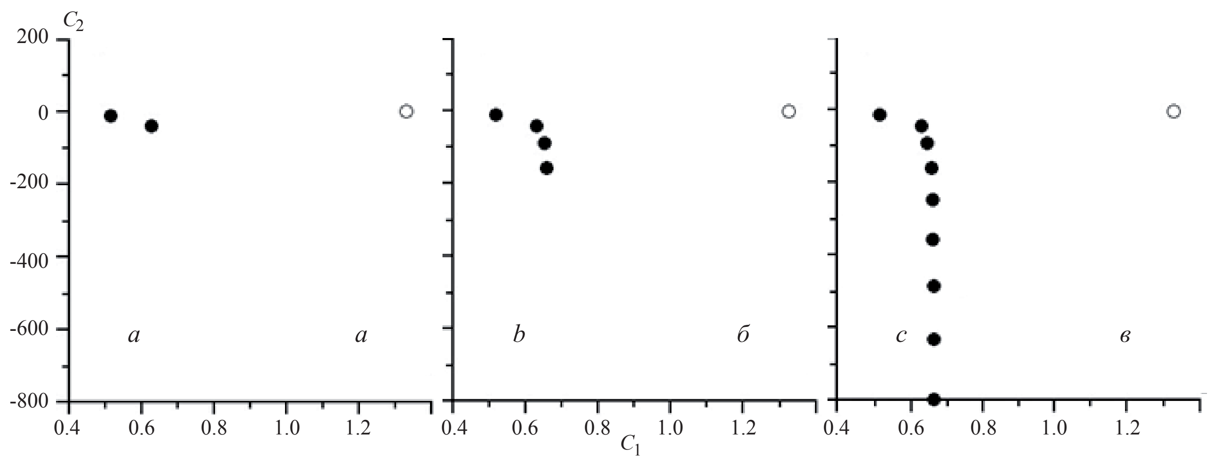

Fig. 1. Calculations of the eigenvalues for $kR = 1$ based on the determination of the polynomials roots.
a — the common roots of the polynomials for $N = 10$ and $N = 20$; *b* — the common roots of the polynomials for $N = 20$ and $N = 40$; *c* — the common roots for $N = 40$ and $N = 50$. The white-filled circle shows the only root with a positive imaginary part (unstable perturbations), the black-filled circles show the roots with a negative imaginary part.

Рис. 1. Расчеты собственных значений задачи при $kR = 1$ на основе определения корней полиномов.
а — общие корни полиномов для $N = 10$ и $N = 20$; *б* — общие корни полиномов для $N = 20$ и $N = 40$; *в* — общие корни полиномов для $N = 40$ и $N = 50$. Белым кружком отмечен единственный корень с положительной мнимой частью (неустойчивые возмущения), черными кружками отмечены корни с отрицательной мнимой частью.

where the coefficients b_i are expressed in terms of the coefficients a_i. This problem can be solved numerically, for example, by means of MATLAB giving $F_{12}(z)$ by series expansion, which consists of a finite number of terms. It is obvious that the more terms in the series, the more accurately eigenvalues and eigenfunctions can be calculated. However, there is one important point to which attention should be paid. Taking a finite number of terms of the series (10), that is assuming that $n_{\max} = N$, the fake roots (not corresponding to the condition $F''_{12}(1) = 0$) will exist among the roots of the polynomial $P(c) = \sum_{n=0}^{n=N} b_n \cdot c^n = 0$. There is a certain technique for filtering out such roots (see, for example, [14]), which in our case reduces to the following. First of all it is required to calculate the roots of the equation $P(c) = \sum_{n=0}^{n=N} b_n \cdot c^n = 0$ for different values of N, considering that as the degree of the polynomial increases the accuracy in determining the mantissa of a number (the number of digits after the decimal point) should be also increased. Then it is necessary to choose from the obtained sets of roots corresponding to different values of N only those roots which are repeated in each set (they can differ only in the accuracy of the mantissa of the number representation).

The results of calculating the roots of the polynomial $P(c) = \sum_{n=0}^{n=N} b_n \cdot c^n = 0$ for different values of N and the parameter $kR = 1$ are shown in fig. 1 in the plane c_1 (axis of abscissae) and c_2 (axis of ordinates).

The common roots of the polynomials for different values of N are shown on fig. 1. As a result of comparing the roots of two polynomials with different values of N, those roots that coincided with an accuracy of at least six decimal digits were considered to be common. The plots show clearly that the number of true roots or eigenvalues of problem (4)—(6) increases with the increase N; these roots are the complex phase velocities of stable perturbations (decreasing with time). Using this method, one can find a sufficiently large number of eigenvalues located along a line almost parallel to the ordinate axis. There is only one root with a positive imaginary part (unstable perturbations), it is marked by a white-filled circle in fig. 1; moreover, this root is the first root and appears already at $N = 2$. Fig. 1 shows that the point on the $c_1 - c_2$ plane corresponding to this root lies far from the main line of points at $kR = 1$ ($kR = 1$ is typical for the frontal zones of the Arctic basin [7]). The similar point arrangement in the complex plane is also typical for $kR = 10$ and $kR = 100$ (see the spectra in [8]). The maximum accuracy of calculating this root using the MATLAB gives the following value: $c_1 = 1.3319139596335208967390841724361$, $c_2 = 0.016783485543458846443095146690677$. This

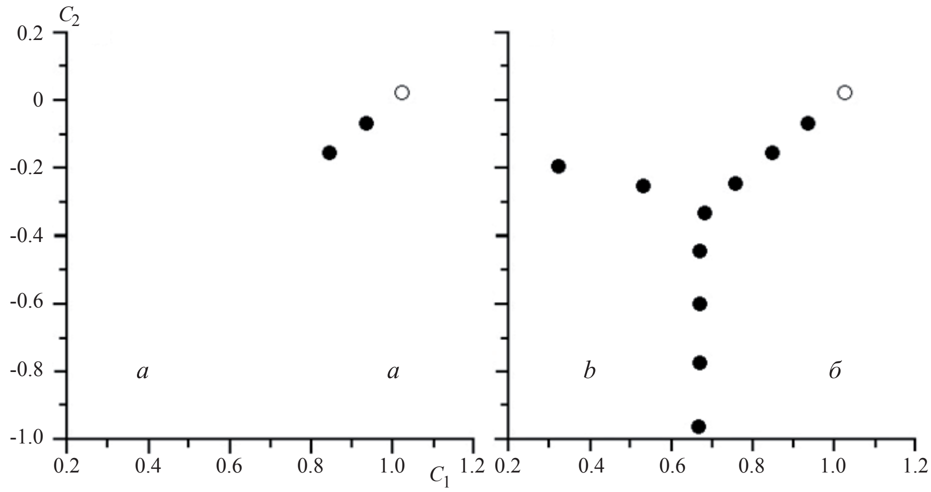

Fig. 2. Calculations of the eigenvalues for $kR = 10^3$ based on the determination of the polynomials roots. *a* — the common roots of the polynomials for $N = 50$ and $N = 100$; *b* — the common roots of the polynomials for $N = 50$ and $N = 200$. The white-filled circle shows the only root with a positive imaginary part (unstable perturbations), the black-filled circles show the roots with a negative imaginary part.

Рис. 2. Расчеты собственных значений задачи при $kR = 10^3$ на основе определения корней полиномов. *а* — общие корни полиномов для $N = 50$ и $N = 100$; *б* — общие корни полиномов для $N = 50$ и $N = 200$. Белым кружком отмечен единственный корень с положительной мнимой частью (неустойчивые возмущения), черными кружками отмечены корни с отрицательной мнимой частью.

value is consistent with accuracy to the 16th decimal digit with the value of this root obtained by another method, which was used for solving of the equivalent problem [11]: c_1 = 1.33191395963352088677, c_2 = 0.01678348554345884397.

Thus the results presented above validate the calculations carried out in [8, 11] and show that the considered simple method is effective for calculating the eigenvalues for not too large values of the problem parameter *kR*.

But it is also interesting to answer the following question: is it possible to calculate the eigenvalues of the problem reliably for large values of the parameter *kR* (i.e. when the highest derivative term in equation (4) or (7) is multiplied by a small parameter) by means of simple method presented above?

The common roots of the polynomial $P(c) = \sum_{n=0}^{n=N} b_n \cdot c^n = 0$ for different values of *N* and $kR = 10^3$ are shown on fig. 2: fig. 2, *a* shows calculation result for $N = 50$ and $N = 100$, fig. 2, *b* — for $N = 100$ and $N = 200$.

The value of the root (or the eigenvalue of the problem) corresponding to the unstable perturbation for $N = 200$ was $c = 1.02236017527081 + 0.02236103467952i$ (where *i* is the imaginary unit). This value is consistent with accuracy to not less than the 11th decimal digit with the eigenvalue of the unstable perturbation for $kR = 10^3$ $c = 1.02236017527 + 0.02236103467i$ presented in [11].

Now two circumstances should be noted. First, according to the presented eigenvalue spectra the phase velocity of stable and unstable perturbations is directed along the flow. However, in contrast to unstable perturbations the phase velocity of stable perturbations is less than the maximum flow velocity. Second, the considered instability of the flow is an oscillatory instability ($c_1 \neq 0$), which is fundamentally different from the typical monotonous instability of fronts on the scales of the intrusion formation, except for the equatorial zone (see [6] for details about monotonous instability).

Summary. This research belongs to the line of studies on instability of geostrophic flow with a linear vertical velocity shear on the scales of intrusion formation [6—8, 11] and is focused on the eigenvalue spectra analysis for stable and unstable perturbations. Before the studies [6—8, 11] the formation of large-scale intrusions observed in the Arctic basin on baroclinic fronts at absolutely stable stratification (decrease of the mean temperature with depth is accompanied by increase of the mean salinity) used to be explained only by interleaving models accounting for the processes of differential mixing [15]. Now the new approaches appear to explain the physical mechanisms of intrusion formation, but they certainly require additional efforts to verify and analyze the obtained results. This work clearly demonstrates the validity of the suggestions made in the papers [6—8, 11], namely: the method for calculation of the eigenvalues of the problem is sufficiently simple in realization, and the results coincide with high accuracy with the calculations presented in [6, 11].

The research was supported by the Russian Science Foundation (project No. 17-77-10080).

Литература

1. *Журбас В. М., Кузьмина Н. П., Кульша О. Б.* Численное моделирование ступенчатого расслоения главного термоклина океана при вырождении термохалинных интрузий солевыми пальцами (численный эксперимент) // Океанология. 1987. Т. 27, № 3. С. 377—383.
2. *Журбас В. М., Кузьмина Н. П., Озмидов Р. В., Голенко Н. Н., Пака В. Т.* О проявлении процесса субдукции в термохалинных полях вертикальной тонкой структуры и горизонтальной мезоструктуры во фронтальной зоне Азорского течения // Океанология. 1993. Т. 33, № 3. С. 321—326.
3. *Walsh D., Polyakov I., Timokhov L., Carmack E.* Thermohaline structure and variability in the eastern Nansen Basin as seen from historical data // Journal of Marine Research. 2007. V. 65, № 5. P. 685—714.
4. *Rudels B., Bjork G., Muench R. D., Schauer U.* Double-diffusive layering in the Eurasian Basin of the Arctic Ocean // J. Mar. Syst. 1999. V. 21, № 1—4. P. 3—27.
5. *Kuzmina N., Rudels B., Zhurbas V., Stipa T.* On the structure and dynamical features of intrusive layering in the Eurasian Basin in the Arctic Ocean // J. Geophys. Res. 2011. V. 116. C00D11. DOI:10.1029/2010JC006920.
6. *Кузьмина Н. П.* Об одной гипотезе образования крупномасштабных интрузий в Арктическом бассейне // Фундаментальная и прикладная гидрофизика. 2016. Т. 9, № 2. С. 15—26.
7. *Kuzmina N. P.* Generation of large-scale intrusions at baroclinic fronts: an analytical consideration with a reference to the Arctic Ocean // Ocean Science. 2016. V. 12. P. 1269—1277. DOI: 10.5194/os-12-1269-2016.
8. *Кузьмина Н. П., Скороходов С. Л., Журбас Н. В., Лыжков Д. А.* О неустойчивости геострофического течения с линейным вертикальным сдвигом скорости в масштабах интрузинного расслоения // Изв. РАН. Физика атмосферы и океана. 2018. Т. 54, № 1. С. 54—63.
9. *Скороходов С. Л.* Численный анализ спектра задачи Орра-Зоммерфельда // Журнал вычислительной математики и математической физики. 2007. Т. 47, № 10. С. 1672—1691.
10. *Скороходов С. Л.* Точки ветвления собственных значений оператора Орра-Зоммерфельда // Доклады Академии Наук. 2007. Т. 416, № 5. С. 600—605.
11. *Скороходов С. Л., Кузьмина Н. П.* Эффективный метод решения модифицированной задачи Орра-Зоммерфельда для анализа неустойчивости течений в Арктическом бассейне // Таврический вестник информатики и математики. 2016. № 3. С. 88—97.
12. *Зайцев В. Ф., Полянин А. Д.* Справочник по обыкновенным дифференциальным уравнениям. М.: Физматлит, 2001. 576 с.
13. *Монин А. С.* Гидродинамическая неустойчивость. Успехи физических наук. 1986. Т. 150. С. 61—105.
14. *Henrici P.* Applied and Computational Complex Analysis. V. 1. New York: Wiley, 1974. 704 p.
15. *Кузьмина Н. П., Журбас Н. В., Емельянов М. В., Пыжевич М. Л.* Применение моделей интерливинга для описания интрузионного расслоения на фронтах глубинной полярной воды Евразийского бассейна (Арктика) // Океанология. 2014. Т. 54, № 5. С. 594—604.