DOI 10.48612/fpg/4eh4–83zr-r1fm

УДК 551.465

© *V. G. Gnevyshev*[1], *T. V. Belonenko*[2]*, 2022
[1]Shirshov Institute of Oceanology, Russian Academy of Sciences, 117997, Nahimovsky Pr., 36, Moscow, Russia
[2]St. Petersburg State University, 199034, 7–9, Universitetskaya Emb., St. Petersburg, Russia
*E-mail: btvlisab@yandex.ru

# ANALYTICAL SOLUTION OF THE RAY EQUATIONS OF HAMILTON FOR ROSSBY WAVES ON STATIONARY SHEAR FLOWS

Received 05.03.2022, Revised 04.06.2022, Accepted 10.06.2022

**Abstract**

The asymptotic behavior of Rossby waves in the ocean interacting with a shear stationary flow is considered. It is shown that there is a qualitative difference between the problems for the zonal and non-zonal background flow. Whereas only one critical layer arises for a zonal flow, then several critical layers can exist for a non-zonal flow. It is established that the integrated ray equations of Hamilton are equivalent to the asymptotic behavior of the Cauchy problem solution. Explicit analytical solutions are obtained for the tracks of Rossby waves as a function of time and initial parameters of the wave disturbance, as well as the magnitude of the shear and angle of inclination of the flow to the zonal direction. The ray equations of Hamilton are analytically integrated for Rossby waves on a shear flow. The obtained explicit expressions make it possible to calculate in real-time the Rossby wave tracks for any initial wave direction and any shear current inclination angle. It is shown qualitatively that these tracks for a non-zonal flow are strongly anisotropic.

**Keywords:** Rossby waves, shear flow, zonal, non-zonal, Hermitian operators, Non-Hermitian operators, ray equations of Hamilton

© *В. Г. Гневышев*[1], *Т. В. Белоненко*[2]*, 2022
[1]Институт океанологии им. П.П. Ширшова РАН, 117997, Нахимовский пр., д. 36, г. Москва, Россия
[2]Санкт-Петербургский государственный университет, 199034, Университетская наб., 7–9, г. Санкт-Петербург, Россия
*E-mail: btvlisab@yandex.ru

# АНАЛИТИЧЕСКОЕ РЕШЕНИЕ ЛУЧЕВЫХ УРАВНЕНИЙ ГАМИЛЬТОНА ДЛЯ ВОЛН РОССБИ НА СТАЦИОНАРНЫХ СДВИГОВЫХ ПОТОКАХ

Статья поступила в редакцию 05.03.2022, после доработки 04.06.2022, принята в печать 10.06.2022

**Аннотация**

Рассматривается асимптотическое поведение волн Россби, взаимодействующих со сдвиговым стационарным течением. Показано, что в этих задачах существует качественное отличие задач для зонального и незонального фонового потока. Если для зонального потока возникает только один критический слой, то для незонального может существовать несколько критических слоев. Установлено, что проинтегрированные лучевые уравнения Гамильтона оказываются равносильны асимптотикам решения задачи Коши. Получены явные аналитические решения для волновых треков волн Россби, как функции времени и начальных параметров волнового возмущения, а также величины сдвига и угла наклона потока к зональному направлению. На примере волн Россби на сдвиговом потоке аналитически проинтегрированы лучевые уравнения Гамильтона. Полученные явные выражения позволяют рассчитывать в реальном времени треки волн Россби для любого начального направления волны и для любого угла наклона сдвигового течения. Показано, что эти треки для незонального потока качественно носят сильно анизотропный характер.

**Ключевые слова:** волны Россби, сдвиговое течение, зональное, незональное, эрмитов оператор, не эрмитов, лучевые уравнения Гамильтона

Ссылка для цитирования: *Гневышев В.Г.*, *Белоненко Т.В.* Аналитическое решение лучевых уравнений Гамильтона для волн Россби на стационарных сдвиговых потоках // Фундаментальная и прикладная гидрофизика. 2022. Т. 15, № 2. С. 8–18. doi:10.48612/fpg/4eh4-83zr-r1fm

For citation: *Gnevyshev V.G.*, *Belonenko T.V. A*nalytical Solution of the Ray Equations of Hamilton for Rossby Waves on Stationary Shear Flows. *Fundamental and Applied Hydrophysics*. 2022, 15, 2, 8–18. doi:10.48612/fpg/4eh4-83zr-r1fm

## 1. Introduction

Historically, the problem of studying the interaction of Rossby waves with large-scale currents began with problems for the atmosphere, in a formulation in which the large-scale background flow was considered strictly zonal [1]. This formulation is quite justified for the atmosphere. Rapid advances in satellite altimetry have contributed to the rapid development of empirical understanding of Rossby waves in the ocean [2]. Analysis of the variability of oceanological fields confirms the existence of Rossby waves in the World Ocean. However, unlike the atmosphere, Rossby waves in the ocean have their specifics. The main difference is that in the ocean, background currents are usually not zonal. Moreover, the strongest dynamic processes occur on non-zonal flows or when the initial zonal flow deviates from the zonal direction as observations show [3, 4].

Critical layers are very important for the understanding of the Rossby waves' interaction with a large-scale flow. The classical critical layer is formally impracticable for Rossby waves. It is the geometric border between the region where the waves exist and the "shadow" area where they do not exist. The critical layer is defined as $c = U$, i.e. the equality of the longitudinal component of the phase velocity of the wave c and the velocity of the background current $U$. The critical layers have been studied and are well known for gravitational waves and internal waves [5–8]. For Rossby waves, the study of the critical layer historically also began with the zonal critical layer. The analysis of Rossby waves on zonal flows is extremely sensitive to the smoothness of the flow and boundary conditions [9].

If the background current is strictly zonal, then, as shown in [3], the determination of the critical layer through the phase velocity is quite correct and can be applied to Rossby waves. However, if the flow is not zonal, such a definition becomes ambiguous and allows Rossby waves to cross the critical layer, with the formation of the so-called overshooting effect. The propagation of Rossby waves on shear flows has its specific feature: the wave track gradually approaches its critical layer, this occurs asymptotically for a long time.

One of the features of Rossby waves is the qualitative difference between the problems for the zonal and non-zonal background flow. The first key point that distinguishes the problems of a zonal background flow and a critical layer from a non-zonal one is the number of critical layers. For a strictly zonal flow, there is only one critical layer, while for a non-zonal shear flow, three qualitatively different cases can be distinguished [3, 4] we will consider a bit later. As a consequence, the passage to the limit from a non-zonal flow to a strictly zonal case is nontrivial. In particular, all asymptotic laws under the passage to the limit are of a discontinuous nature [3, 4]. In this case, of the three non-zonal critical layers in the passage to the limit, from the non-zonal to the zonal critical layer, only one critical layer remains. And the transition from the zonal to the non-zonal case, in principle, is not possible. As a consequence, a strictly meridional flow acquires the most general character, rather than a purely zonal flow.

The second important point for Rossby waves is that the linear operator of Rossby waves ceases to be Hermitian upon passing to the non-zonal case. The adiabatic invariant in the form of the enstrophy conservation law, which exists in the WKB approximation, ceases to hold for non-zonal piecewise linear flow profiles of the "vortex layer" type. A non-zonal strong shear current enters into an active exchange of vorticity with Rossby waves [5, 10–13].

The fundamental point in which the analysis of problems for the ocean differs from the atmosphere is the limitedness of ocean currents in space and, as a consequence, in time. Therefore, for the obtained qualitative results of the analysis of the dynamics of Rossby waves to have an applied character, it is important to understand what periods and spatial scales are behind such concepts as "approaching" the critical layer?

The classical approach for analyzing the kinematics of waves in dispersive systems is based on the ray equations of Hamilton. However, as is customary even in classical mechanics, no one explicitly solves the differential equations of Hamilton in analytical form. The traditional approach is qualitative and is based on the presence of cyclic variables in the problem. As a rule, these are the longitudinal component of the impulse and the frequency of the wave. If we also use a certain set of symmetries, related to the Hermitian nature of the linear wave operator, then this purely geometric approach suffices to understand qualitatively the evolution of waves on plane-parallel inhomogeneous flows, without solving the ray equations of Hamilton explicitly. Therefore, it is better to use a qualitative method, which is called the isofrequency method. It is based on the geometric construction of isofrequency lines and the concept of the direction of the group velocity. For Rossby waves, a qualitative analysis of the kinematics based on the isofrequency method was performed as early as [14, 15].

Based on the fact that asymptotically long adhesion of Rossby waves to the critical layer has already been established, we are trying to understand the specific features of this process. The goal of our work is to determine how real the periods and spatial scales of this process are so that they can be realized for real conditions in the ocean. To answer this question, it is necessary to have explicit analytical solutions for wave tracks as a function of time and initial parameters of the wave disturbance, as well as the magnitude of the shear and the angle of inclination of the flow to the zonal direction. In addition, in this paper, using the example of Rossby waves on a shear flow, we analytically integrate the ray equations of Hamilton for the first time. The obtained explicit expressions make it possible to calculate in real-time the Rossby wave tracks for any initial wave direction and any shear current inclination angle. As will be shown below, such tracks for a non-zonal flow are qualitatively highly anisotropic.

The generally accepted way to obtain a solution as a function of the initial position of the wave and time is to solve the Cauchy problem. For barotropic Rossby waves, the Cauchy problem was solved in [16, 17] for strictly zonal and meridional currents. Continuing this direction, we will show that the integrated ray equations of Hamilton turn out to be equivalent to the asymptotics of the solution of the Cauchy problem. However, in contrast to [16, 17], we propose an easier way to obtain explicit analytical expressions for the Rossby wave tracks. To obtain a solution, the introduction of convective coordinates, direct and inverse Fourier transforms, and the stationary phase method for the obtained two-dimensional Fourier integral is not required [16, 17]. In this work, we will show that ray equations of Hamilton for Rossby waves are integrated with explicit expressions quite simply using the arctangent and logarithm functions, in contrast to the solutions of [16, 17], which use a more specific mathematical apparatus related to the Cauchy problem. The new solutions of the ray equations of Hamilton for Rossby waves are much simpler than the geometric method of isofrequencies and represent explicit analytical expressions for the tracks of Rossby waves in elementary functions.

## 2. Methods

The ray equations of Hamilton are an effective tool for analyzing the kinematic properties of Rossby waves in a plane-parallel shear flow [5, 18]. In practice, this method is often successfully applied in numerical calculations (see, for example, [19]). We will show that for shear flows there is also an explicit analytical solution to these equations, and these solutions will be found in elementary functions. The so-called equations of geometric optics are as follows:

$$k_t = -\frac{\partial \omega}{\partial X}, \quad l_t = -\frac{\partial \omega}{\partial Y}, \tag{1}$$

$$X_t = \frac{\partial \omega}{\partial k}, \quad Y_t = \frac{\partial \omega}{\partial l}. \tag{2}$$

Here $x$ and $y$ are the axes of the Cartesian coordinate system directed to the east and north, respectively; $t$ is the time; $(k, l)$ are the components of the wave vector $\kappa$, $\omega$ is the frequency, $X = X(\omega, k, l)$ and $Y = Y(\omega, k, l)$ are the ray variables in a coordinate system rotated counterclockwise by an angle $\theta$.

Let us assume that the background flow is a stationary shear flow directed at a certain angle $\theta$ fixed to the parallel. For certainty, we will consider the angle the angle $\theta > 0$ if it is counted counterclockwise. To find a solution, we will proceed as follows. At the first stage, let us go over to the coordinate system associated with the flow. Then in the new coordinate system rotated by the angle $\theta$, the background current velocity field has only one longitudinal velocity component $\vec{U} = (U, 0) = (U(y), 0)$. Further, the coordinate system is chosen so that at its origin the velocity field is zero. Assume that $U$ is approximately linear in $y$: $U = U_y y$. Having solved the problem in a new (rotated) coordinate system, we then make a reverse rotation by an angle $(-\theta)$, and thus we get a solution in the original coordinate system tied to the parallel and the meridian, which is more convenient for a clear illustration of the result.

The dispersion relation in the new coordinate system is

$$\omega = -\frac{\beta(k\cos\theta - l\sin\theta)}{k^2 + l^2 + F^2} + kU_y y, \tag{3}$$

where $\beta = \frac{df}{dy}$, $f$ is the Coriolis parameter, $F^2 = \frac{f^2}{gH}$, g is the acceleration of gravity, $H$ is the depth of the ocean [3, 21]. In the new coordinate system, there are two cyclic variables; they are the longitudinal coordinate $x$ and time $t$. Consequently, the problem has two integrals of motion: the longitudinal component of the momentum (in the ray approach, this is the $x$-component of the wavenumber $\kappa$) and the wave frequency $\omega$.

The integrated first pair of equations (1) has the form:

$$k = k_0 = \text{const}, \; l_c = l_0 - U_y k_0 t, \tag{4}$$

where $(k_0, l_0)$ are the initial components of the wavenumber at $t = 0$. Note that the integrated first pair of the equations of Hamilton gives a result that is identical to the result obtained in the framework of the Cauchy problem [3, 21].

Integrating Eqs. (2), we find the coordinates of the quasi-monochromatic wave packet, at the initial moment located at the origin of coordinates:

$$Y_\theta = \frac{\beta}{U_y}\left[\frac{\cos\theta - \sin\theta\left(\frac{l_0}{k_0} - U_y t\right)}{k_0^2 + F^2 + \left(l_0 - k_0 U_y t\right)^2} - \frac{\cos\theta - \sin\theta\left(\frac{l_0}{k_0}\right)}{k_0^2 + F^2 + l_0^2}\right], \tag{5}$$

$$X_\theta = \frac{\beta\cos\theta}{U_y}\left[\frac{k_0}{k_c^3}\left\{-\arctan\left(\frac{l_c}{k_c}\right) + \arctan\left(\frac{l_0}{k_c}\right)\right\}\right] - \frac{\beta\cos\theta}{U_y k_c^2}\left[\frac{F^2 U_y t + k_0 l_0}{l_c^2 + k_c^2} - \frac{k_0 l_0}{l_0^2 + k_c^2}\right] +$$
$$+ \frac{\beta\sin\theta}{U_y}\left[\frac{1}{2k_0^2}\ln\left(\frac{l_c^2 + k_c^2}{l_0^2 + k_c^2}\right) - \frac{1 - U_y t l_c k_0^{-1}}{l_c^2 + k_c^2} + \frac{1}{l_0^2 + k_c^2}\right] + U_y t Y_c. \tag{6}$$

The subscript index θ in the solution ($X_0$, $Y_0$) shows that this solution was found in a coordinate system rotated counterclockwise by an angle θ. For simplicity, the following notation is introduced in formula (6):

$$l_c = l_0 - U_y k_0 t,\ k_c = \sqrt{k_0^2 + F^2}. \tag{7}$$

Let us turn to dimensionless variables taking into account the Rossby baroclinic radius: $k^* = k_0/F$, $l^* = l_0/F$, $k_c^* = k_c / F$, $l_c^* = l_c / F$, $X^* = X_c F$, $Y^* = Y_c F$, and dimensionless time for the shear of the background flow velocity: $t^* = t|U_y|$. Omitting the asterisks, we get:

$$Y_\theta = \beta_0\left[\frac{\cos\theta - l_c k^{-1}\sin\theta}{k_c^2 + l_c^2} - \frac{\cos\theta - l k^{-1}\sin\theta}{k_c^2 + l^2}\right], \tag{8}$$

$$X_\theta = \beta_0\cos\theta\left[\frac{k}{k_c^3}\left\{-\arctan\left(\frac{l_c}{k_c}\right) + \arctan\left(\frac{l}{k_c}\right)\right\}\right] - \frac{\beta_0\cos\theta}{k_c^2}\left[\frac{k\,l + t}{l_c^2 + k_c^2} - \frac{k\,l}{l^2 + k_c^2}\right] +$$
$$+ \beta_0\sin\theta\left[\frac{1}{2k^2}\ln\left(\frac{l_c^2 + k_c^2}{l^2 + k_c^2}\right) - \frac{1 - t l_c k^{-1}}{l_c^2 + k_c^2} + \frac{1}{l^2 + k_c^2}\right] + tY, \tag{9}$$

where

$$l_c = l - kt,\ k_c = \sqrt{k^2 + 1},\ U_y > 0,\ \beta_0 = \frac{\beta}{FU_y}, \tag{10}$$

if $t$ replaced with $-t$

$$U_y > 0. \tag{11}$$

This solution can be simply represented as:

$$X_\theta = X_1\cos\theta + X_2\sin\theta,\ Y_\theta = Y_1\cos\theta + Y_2\sin\theta, \tag{12}$$

where ($X_1$, $Y_1$) is the packet coordinates in the case when the flow is zonal (directed along the parallel: $\theta = 0$), and ($X_2$, $Y_2$) is the packet coordinates in the case when the flow is meridional (directed along the meridian). It is important to note that $\theta = \frac{\pi}{2}$ for the meridional direction and the $OX_1$ axis is directed to the north and the $OX_2$ is to the west.

$$X_1 = \frac{\beta_0 k}{k_c^3}\left[-\arctan\left(\frac{l_c}{k_c}\right) + \arctan\left(\frac{l}{k_c}\right)\right] - \frac{\beta_0}{k_c^2}\left[\frac{k\,l + t}{l_c^2 + k_c^2} - \frac{k\,l}{l^2 + k_c^2}\right] + tY_1, \tag{13}$$

$$Y_1 = \beta_0\left[\frac{1}{k_c^2 + l_c^2} - \frac{1}{k_c^2 + l^2}\right], \tag{14}$$

$$X_2 = \beta_0\left[\frac{1}{2k^2}\ln\left(\frac{l_c^2 + k_c^2}{l^2 + k_c^2}\right) - \frac{1 - t l_c k^{-1}}{l_c^2 + k_c^2} + \frac{1}{l^2 + k_c^2}\right] + tY_2, \tag{15}$$

$$Y_2 = \beta_0\left[\frac{l k^{-1}}{l^2 + k_c^2} - \frac{l_c k^{-1}}{l_c^2 + k_c^2}\right]. \tag{16}$$

Then, designating the coordinates of the package in the coordinate system tied to the east and north directions ($X$, $Y$), you need to reverse the rotation of the coordinate system (counterclockwise). Finally, we get the following expressions in a matrix form:

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos\theta, & -\sin\theta \\ \sin\theta, & \cos\theta \end{pmatrix} \begin{pmatrix} X_\theta \\ Y_\theta \end{pmatrix} \quad (17)$$

or

$$X = X_1 \cos^2\theta + \left(X_2 - Y_1\right)\cos\theta\sin\theta - Y_2 \sin^2\theta, \quad (18)$$

$$Y = Y_1 \cos^2\theta + \left(X_1 + Y_2\right)\cos\theta\sin\theta + X_2 \sin^2\theta. \quad (19)$$

## 3. Results

### *3.1. Numerical estimation of dimensionless parameters*

We will take as the initial the following characteristic physical scales for the ocean: $f = 10^{-4}$ s$^{-1}$, $\beta = 10^{-11}$ m$^{-1}$ s$^{-1}$, $F = 0.5 \times 10^{-5}$ m$^{-1}$. Some numerical estimates give something like this: whereas we take for the scale of the background flow velocity $U = 5$ cm/sec., and the scale of the background flow variability 50 km, then the unit of the dimensionless time scale $U_y^{-1}$ is about 11 days. Therefore, the dimensionless time $t = 2.86 \times \pi$ is about 3 months. In this case, the dimensionless parameter $\beta_0$ is equal to 0.5. If we take 100 km as the scale of the background flow variability, then the unit of the dimensionless time scale $U_y^{-1}$ is approximately 22 days. Then the dimensionless time $t = 2.86 \times \pi$ (this corresponds to 6 months in dimensional units), and the dimensionless parameter $\beta_0$ is equal to 1.0. These estimates make the results obtained physically justified and correct for practical use.

### *3.2. Graph analysis*

Qualitatively, all plots can be divided into two cases: a zonal (Fig. 1) and non-zonal flow (Fig. 2). A common property of all graphs is that with increasing time, all rays adhere to the critical layer. However, the number of critical layers, as well as their location, is a nontrivial function of the angle of inclination of the background flow. Qualitatively, several main scenarios can be distinguished.

*Zonal flow scenario.* If the flow is strictly zonal, $\theta = 0$ (Fig. 1), then one critical layer is formed, which does not depend on the initial direction of the group velocity and is determined only by the magnitude of the modulus of the initial wavenumber. The expression for the ordinate of the critical layer is determined by the following (nonzero) value:

$$Y_{1c}\Big|_{t\to\infty} \to -\beta_0 \left(k_c^2 + l_0^2\right)^{-1}. \quad (20)$$

In the case of a strictly zonal flow, all waves adhering to the critical layer move strictly to the west: $X_{1c}\Big|_{t\to\infty} \to -\infty$. The explanation of the term “adhering” is given by [3]. It is also important to note that the movement of Rossby waves at certain points in time is possible both to the east and in other directions. However, with increasing $t$, all rays adhere to the critical layer, moving strictly to the west. An analysis of the tracks shows that the dimensionless values at which the movement begins to follow a strictly westerly direction is approximately $t = 8$, and it gives a period of about three months for the open ocean.

In the case of a zonal flow, the initial component of the group velocity in the meridional direction is proportional to $k_0 \times l_0$. For the zonal component of the group velocity, the sign is determined by the following expression: $\left(k_0^2 - l_0^2 - 1\right)$. To have an idea of all possible cases, it suffices to take the following set of four initial wavenumbers ($k_0$, $l_0$). Figure 1 shows four options for the initial direction of the group velocity; the tracks are drawn for the case $U_y > 0$. The abscissa axis is directed to the east, the ordinate is to the north. Track 1 — the initial group velocity is directed to the southwest. The initial components of the wavenumber are $k_0 = -1$, $l_0 = 1$. Track 2 — the initial group velocity is directed to the southeast: $k_0 = -4\sqrt{2}/\sqrt{17}$, $l_0 = \sqrt{2}/\sqrt{17}$ or $k_0 = -1.372$, $l_0 = 0.343$. Track 3 — the initial group velocity is directed to the north-east: $k_0 = -4\sqrt{2}/\sqrt{17}$, $l_0 = -\sqrt{2}/\sqrt{17}$ or $k_0 = -1.372$, $l_0 = -0.343$. Track 4 — the initial group velocity is directed to the northwest: $k_0 = -1$, $l_0 = -1$. The wavenumbers are specially selected so that the tracks adhere to one critical layer. For all four combinations, the relation $k_0^2 + l_0^2 + 1 = 3$.

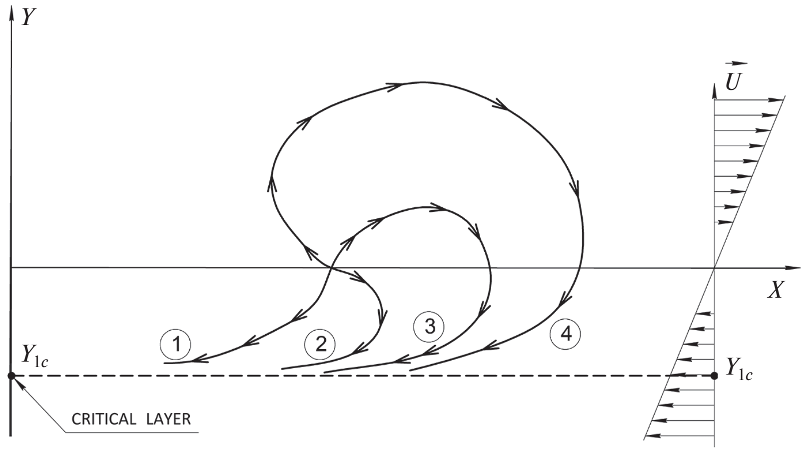


**Fig. 1**. The variety of tracks of Rossby waves in their interaction with the zonal flow. Descriptions of tracks 1–5 are given in the text

*Non-zonal flow scenario*. For a strictly meridional flow $\theta = \frac{\pi}{2}$, there are three qualitatively different cases for the implementation of the critical layer, which can be conventionally called "positive", "negative" and "zero". For the case of a strictly zonal flow, the critical layer is the boundary of the region where the waves exist. For any non-zonal flow, additional critical layers appear that are inside this region. The critical layer is "negative" when the sign of the intrinsic frequency adhering to the critical layer is negative. Such waves with a negative intrinsic frequency are commonly called "waves of negative energy" [11]. The peculiarity of the non-zonal case is that Rossby waves, starting from zero value, can change the sign of their intrinsic frequency at a certain moment in time.

The expression for the ordinate of the critical layer is determined by the following value.

$$Y_{2c}\big|_{t\to\infty} \to \frac{l_0}{k_0}\left[\frac{\beta_0}{k_c^2 + l_0^2}\right]. \tag{21}$$

Recall that the coordinate system is tied to the direction of the flow velocity, so in this case, when $\theta = \frac{\pi}{2}$, the $x$-axis is directed to the north and the $y$-axis to the west.

Group speed signs are defined as follows:

$$C_{grx} \approx \left(-kl\right), \quad C_{gry} \approx \left(k^2 + 1 - l^2\right).$$

Consider the case $U_y > 0$. Provided $lk^{-1} > 0$, waves adhere to the negative critical layer: ($Y_{2c} > 0$). Wherein $X_{2c}\big|_{t\to\infty} \to -\infty$, and the value of the group velocity along the $x$-coordinate turns out to be negative. That is, it turns out that for adhesion to the negative critical layer, the wave must start against the direction of the flow, but the flow will certainly turn the wave in the direction of the flow. The wave will cross the critical layer, change the sign of its intrinsic frequency, reflect from the higher value of the background flow velocity, and start again approaching the critical layer, but from the opposite side. This wave behavior is called overshooting (see [3]); it also occurs in quantum mechanics.

For the initial values ($k_0 = 1$, $l_0 = 1$), the direction of the group velocity has the opposite direction with respect to the flow, and a negative critical layer is realized. Whereas for the initial values ($k_0 = -1$, $l_0 = 1$), the direction of the group velocity coincides with the direction of the flow, and the negative critical layer is not realized. Reflection occurs, and the wave goes to the positive critical layer.

Provided $l_0 k_0^{-1} < 0$, waves adhere to the positive critical layer, ($Y_{2c} < 0$). The situation is qualitatively similar to the purely zonal case. In this case, the critical layers have not only components of different signs and magnitude, but also tend to $\pm\infty$ by the $x$-coordinate, ($X_{2c} \to -\infty$).

From the analysis of these ratios, it can be seen that an additional second critical layer appears due to the non-zoning of the flow. It is realized only for waves that initially propagate strictly against the current. The waves moving in the direction of the flow have a trivial reflection from the negative critical layer. Let us also note the existence of a third scenario. At $l_0 = 0$, the wave starts strictly perpendicular to the background current, while the critical layer ($Y_{2c} = 0$) is zero.

Let us analyze the intermediate flow direction. The asymptotics for the ordinate of the critical layer in the general case has the form:

$$Y_\theta\big|_{t\to\infty} \to \frac{l_0\sin\theta - k_0\cos\theta}{k_0}\left[\frac{\beta_0}{k_0^2+l_0^2}\right]. \tag{22}$$

The longitudinal component of the group velocity is proportional to

$$\left(k_0^2 - l_0^2 - 1\right)\cos\theta - 2k_0 l_0 \sin\theta.$$

The transverse component of the group velocity is proportional to

$$2k_0 l_0 \cos\theta - \left(l_0^2 - k_0^2 - 1\right)\sin\theta.$$

It follows from expression (22) that when even weak non-zonality appears, there is not one, as in the case of a purely zonal flow, but three critical layers since the value ($l_0\sin\theta - k_0\cos\theta$) can be positive or negative values or zero. For zonal flow, regardless of the parameters of the wavenumber of the incident wave, any wavenumbers can be considered, however, the critical layer is always at negative velocities. For a non-zonal flow at different wavenumbers, that is, at different angles of incidence on the flow, there will be three such critical layers: one at a negative velocity value, one at a positive velocity value, and one with zero velocity. If we fix the wavenumber, then there is always one critical layer. For a zonal flow, this layer will correspond to a negative velocity value. For non-zonal flow, there are possible options: the critical layer will be located either at a positive velocity value or at a negative one, or with zero velocity. In other words, some wavenumbers will stick to the positive, and others to the negative values of the background velocity. When we say "one critical layer", we do not mean a fixed value of the velocity, but only its sign.

The first critical layer that is implemented for western propagation is the classic well-known and well-studied critical layer for Hermitian operators. The second critical layer is realized for waves moving eastward. This critical layer does not have symmetries due to the non-Hermitian nature of the non-zonal linear operator of Rossby waves and introduces such a phenomenon as overshooting into the kinematics of Rossby waves. The third critical is zero and is inherent only in strictly non-zonal flows. In this scenario, the waves return to the initial level from which they started.

For simplicity of numerical values, we take the angle $\theta = \frac{\pi}{4}$. Then we have the following typical sets of wave tracks: track 1 — ($k_0 = -0.5$, $l_0 = 1$); track 2 — ($k_0 = -1$, $l_0 = 1$); track 3 — ($k_0 = -2$, $l_0 = -0.5$); track 4 — ($k_0 = -1$, $l_0 = -2$); track 5 — ($k_0 = -1$, $l_0 = -1$). Such a variety of possible scenarios is typical for Rossby waves and is associated with the absence of symmetries in the problem, which are a consequence of the non-Hermitian nature of the linear operator of Rossby waves for arbitrary shear flows.

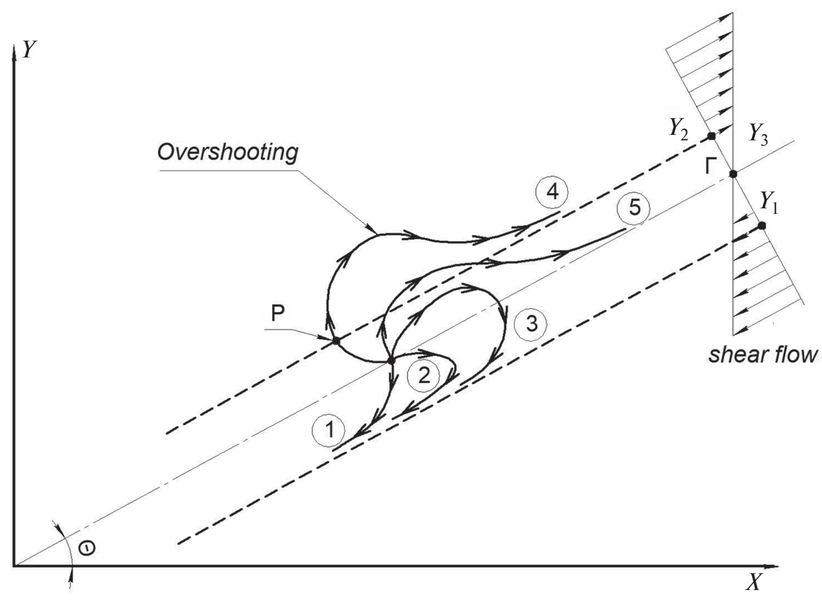


**Fig. 2**. The variety of Rossby wave tracks in their interaction with the non-zonal current. The description of tracks 1–4 is given in the text

The ray equations of Hamilton are a kind of approximate method for analyzing the kinematics of waves. Therefore, a question arises: what are the limits of applicability of these equations?

To answer this question, we will proceed with the statement that, from a mathematical point of view, the solution of the Cauchy problem is more correct than the ray equations of Hamilton. The solution of the Cauchy Problem for Rossby waves on a shear plane-parallel flow, in a coordinate system associated with the flow and directed at a certain angle θ to parallel, has the form [3, 4]:

$$\Psi(x,y,t)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(k,l)\frac{\left(k_z^2+l^2\right)}{\left(k_z^2+l_t^2\right)}\times\exp\left(i\Upsilon(x,y,k,t)\right)dk\,dl,$$

$$\Upsilon(x,y,k,l,t)\equiv\frac{\beta\cos\theta}{U_y k_z}\left\{-\arctan\left(\frac{l_t}{k_z}\right)+\arctan\left(\frac{l}{k_z}\right)\right\}+\frac{\beta\sin\theta}{2U_y k}\ln\left(\frac{k_z^2+l_t^2}{k_z^2+l^2}\right)+\left[k\left(x-U_y yt\right)+ly\right], \tag{23}$$

where the following designations are introduced: $l_t = l - U_y kt,\ k_z=\sqrt{k^2+F^2}$. We construct the phase for the solution in the form of the ray equations as follows:

$$\Theta(x,y,k,l,t)=-\int\omega dt. \tag{24}$$

Let us substitute in (24) the expression for the frequency (3) and the first pair of integrated equations (4). In this case, using the free term in the form of an arbitrary function of the wavenumbers, we normalize the phase as follows: $\Theta(y,k,l,t)\big|_{t=0}=0.$ Integrating (24) with the chosen normalization conditions, we obtain:

$$\Theta(y,k,l,t)=-\int\left\{\frac{-\beta\left(\kappa_0\cos\theta-l_c\sin\theta\right)}{\kappa_0^2+l_c^2+F^2}+\kappa_0 U_y y\right\}dt=$$

$$=\frac{\beta\cos\theta}{U_y\kappa_z}\left\{-\arctan\left(\frac{l_c}{\kappa_c}\right)+\arctan\left(\frac{l_0}{\kappa_c}\right)\right\}+\frac{\beta\sin\theta}{2U_y\kappa_0}\ln\left(\frac{\kappa_c^2+l_c^2}{\kappa_c^2+l^2}\right)-\kappa_0 U_y yt\,. \tag{25}$$

Comparing the obtained expression (25) for the normalized phase of the WKB-solution with the expression for the phase of solution (23) of the Cauchy problem, we find the following relation:

$$\Theta(y,k,l,t)+kx+ly=\Upsilon(x,y,k,l,t).$$

Thus, the phases of the solutions coincide. On the other hand, if we assume that the scale of changes in the main flow is much larger than the characteristic scale of the solution for perturbations, then a small parameter ε will appear in the problem [20, 21], which formally, after reduction to dimensionless form, is expressed by replacing the derivative the main flow velocity $U_y$ by $\varepsilon\times U_y$. Passing in the expression for the phase of solution (23) to the limit in $U_y$, as in a small parameter, and keeping the zero and first terms of the expansion, we obtain the following relation:

$$\Upsilon(x,y,k,l,t)_{\left(U_y t\to 0\right)}\to\left(\frac{-\beta\left(\kappa\cos\theta-l\sin\theta\right)}{\kappa^2+l^2+F^2}+\kappa U\right)t+\kappa x+ly=\omega t+\kappa x+ly,$$

$$\text{where }\ \omega=\frac{-\beta\left(\kappa\cos\theta-l\sin\theta\right)}{\kappa^2+l^2+F^2}+\kappa U.$$

On the other hand, from (23) it is easy to obtain the following relation:

$$\lim\Upsilon(x,y,k,l,t)_{\left(U_y t\to\infty\right)}\neq\omega\,t+\kappa\,x+l\,y.$$

**4. Discussion and Conclusions**

Summing up, let us emphasize the first original result obtained in this work. Solutions (5) and (6) obtained in the framework of the Cauchy problem are exact solutions of ray equations (1) and (2). Consequently, not only do the limiting values obtained within the framework of the WKB-solution and the Cauchy problem in the first approximation coincide, but also the solutions themselves. In other words, the integral of the solution phase, obtained in the first order of the WKB approximation and normalized to zero at the initial moment, coincides with the phase of the basic solution of the Cauchy problem. In this case, the expansion of the phase of the solution to the Cauchy problem in terms of the small WKB-parameter in the first approximation gives the dispersion relation obtained in the first order of the WKB-solution. For large time intervals, the phase of the solution to the Cauchy problem does not reach the WKB-solution mode. Hence, from the point of view of the Cauchy problem, the WKB-solution cannot work up to any infinitely large times with a finite shear of the background flow velocity profile.

Otherwise, it can be explained as follows. The time $t$ and the shear of the background current velocity $U_y$ are included in the solution in the form of the product $t \times U_y$. Consequently, whatever the small parameter $U_y$, there will come a time t such that the product $t \times U_y$ will be greater than one, and the series expansion of the solution phase will no longer be justified.

Thus, the application of the Hamiltonian formalism in a linear problem helps to build a bridge between seemingly different solutions obtained in the WKB-approximation and the framework of the Cauchy problem. In this case, the first pair of ray equations (1) is nothing but the condition of equality of the cross derivatives of the solution phase. The second pair of ray equations (2) is the equation for a stationary point. The mathematical reason for this behavior is that in the presence of non-zoning in the solution phase, a logarithm of the form appears $\ln\left(1+U_y^2t^2\right)$. The Taylor series of the logarithm at zero has a radius of convergence equal to one. Consequently, no matter how small the value of the shear in the profile of the background flux $U_y$ is, there will come a time at which the argument of the logarithm will exceed one and the asymptotic expansion will stop working.

In this paper, using the example of Rossby waves on non-zonal shear flows, explicit analytical integration of the ray equations of Hamilton is performed for the first time. Previously, no one paid attention to this possibility. It turned out that the obtained explicit analytical solution of ray equations of Hamilton is expressed in simple elementary functions, which turned out to be quite unexpected. The constructed typical kinematic tracks of Rossby waves on non-zonal shear currents show the relevance of such a phenomenon as the critical layers of Rossby waves.

In its simplicity and ease, this method surpasses the solution in terms of the Cauchy problem using convective coordinates, and from an analytical point of view, it is identical to the asymptotics of the two-dimensional integral of the Cauchy problem that we obtained earlier [3].

An analytical comparison of the obtained solution with the solution of the Cauchy problem for Rossby waves is made. For small time intervals, the solutions of the ray equations strictly coincide with the asymptotics of the integral obtained in the framework of the Cauchy problem. The non-zonality of the flow leads to the appearance of a logarithm in the solution phase, which greatly complicates the convergence of the results obtained. At large time intervals, the non-zonality of the flow leads to a logarithmic spreading of the solution, which requires additional analysis within the framework of the convolution of the obtained solutions over the spectrum of wavenumbers.

The obtained analytical expressions were used to construct the kinematic tracks of Rossby waves on shear flows. The solutions are anisotropic and, in the general case, do not have classical north-south symmetries.

It is shown that in the non-zonal case, a second critical layer is added to the classical critical layer of Rossby waves for the strictly zonal case, which is directly related to such concepts as negative energy waves and overshooting.

**5. Funding**

The publication was funded by the Russian Science Foundation, project No 22-27-00004. The work of V.G.G. was carried out within the State Task for the Shirshov Institute of Oceanology RAS, project No 0128-2021-0003.

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